

THE CIRCLE METHOD AND NON LACUNARITY OF MODULAR FUNCTIONS

SANOLI GUN AND JOSEPH OESTERLÉ

ABSTRACT. Serre proved that any holomorphic cusp form of weight one for $\Gamma_1(N)$ is lacunary while a holomorphic modular form for $\Gamma_1(N)$ of higher integer weight is lacunary if and only if it is a linear combination of cusp forms of CM-type (see [5], subsections 7.6 and 7.7). In this paper, we show that when a non-zero modular function of arbitrary real weight for any finite index subgroup of the modular group $\mathbf{SL}_2(\mathbf{Z})$ is lacunary, it is necessarily holomorphic on the upper-half plane, finite at the cusps and has non-negative weight.

INTRODUCTION

Let g be a holomorphic function on some half plane $\mathfrak{H}_r = \{z \in \mathbf{C} \mid \text{Im}(z) > r\}$ with $r \in \mathbf{R}$. Assume that there exists some real number $h > 0$ and some complex number u such that $|u| = 1$ and $g(z+h) = ug(z)$ for $z \in \mathfrak{H}_r$. Then g has a unique expansion of the form

$$(1) \quad g(z) = \sum_{\lambda \in \Lambda} a(\lambda) e^{2\pi i \lambda z},$$

where Λ consists of all $\lambda \in \mathbf{R}$ such that $e^{2\pi i \lambda h} = u$. We have $\Lambda = \lambda_0 + \mathbf{Z}h^{-1}$ for any $\lambda_0 \in \Lambda$. The series (1) is called the *expansion of g at ∞* . It converges normally on any horizontal strip $\{z \in \mathbf{C} \mid r' < \text{Im}(z) < r''\}$ with $r < r' < r''$. For $\lambda \in \Lambda$, the coefficient $a(\lambda)$ is given by

$$(2) \quad a(\lambda) = h^{-1} \int_{[\tau, \tau+h]} g(z) e^{-2\pi i \lambda z} dz$$

where τ is any point of \mathfrak{H}_r . We shall say that:

- g has *finite order at ∞* if $\{\lambda < 0 \mid a(\lambda) \neq 0\}$ is finite;
- g is *finite at ∞* if $a(\lambda) = 0$ for all $\lambda < 0$; in that case (1) converges normally on any half plane $\mathfrak{H}_{r'}$ with $r' > r$.

If g has finite order at ∞ , we shall say that its expansion at ∞ :

- is *lacunary* if $\#\{\lambda \leq x \mid a(\lambda) \neq 0\} = o(x)$ when $x \rightarrow +\infty$;

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– is *strongly non-lacunary* if there exists a non-constant arithmetic progression $\Lambda' = \lambda_0 + mh^{-1}\mathbf{N}$ contained in Λ such that $a(\lambda) \neq 0$ for all $\lambda \in \Lambda'$; in that case the expansion of g at ∞ is not lacunary.

All these definitions depend only on g and not on the choice of h and u .

We now consider a finite index subgroup Γ of $\mathbf{SL}_2(\mathbf{Z})$ and a real number k . We define z^k for $z \in \mathbf{C}^*$ to be $e^{k \log z}$, where the logarithm of z is chosen with imaginary part in $] -\pi, \pi]$.

Definition 1.— We say that a meromorphic function f on the upper half plane $\mathfrak{H} = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$ is a modular function of weight k for Γ if:

a) for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, f satisfies a functional equation of the form $f(\gamma z) = j(\gamma, z)f(z)$, where the automorphy factor $j(\gamma, z)$ is a holomorphic function on \mathfrak{H} such that $|j(\gamma, z)| = |cz + d|^k$ and where $\gamma z = \frac{az+b}{cz+d}$;

b) for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$, the function $z \mapsto (cz+d)^{-k} f(\frac{az+b}{cz+d})$ is holomorphic on some half plane \mathfrak{H}_r (hence by a) satisfies the conditions stated for g at the beginning of this introduction) and it has finite order at ∞ .

Note that when $f \neq 0$, condition a) determines the automorphy factors $j(\gamma, z)$. They satisfy the cocycle relation

$$(3) \quad j(\gamma\gamma', z) = j(\gamma, \gamma'z)j(\gamma', z)$$

for $\gamma, \gamma' \in \Gamma$ and $z \in \mathfrak{H}$. Condition b) is then expressed by saying that f has finite order at the cusps; if moreover $z \mapsto (cz+d)^{-k} f(\frac{az+b}{cz+d})$ is finite at ∞ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$, then f is said to be finite at the cusps.

Definition 2.— A modular function is said to be lacunary (resp. strongly non-lacunary) if its expansion at ∞ is lacunary (resp. strongly non-lacunary).

We can now state the main theorems of this paper. Let k be a real number and Γ be a finite index subgroup of $\mathbf{SL}_2(\mathbf{Z})$. Let f be a non-zero modular function of weight k for Γ , with respect to some automorphy factors.

Theorem 1.— If f is not holomorphic on \mathfrak{H} , f is not lacunary.

From now on, we assume f to be holomorphic on \mathfrak{H} .

Theorem 2.— If $k < 0$, f is strongly non-lacunary.

Theorem 3.— If $k \geq 0$ and f is not finite at the cusps, f is strongly non-lacunary.

In the case $\Gamma = \mathbf{SL}_2(\mathbf{Z})$, one can be more precise. There exists $u \in \mathbf{C}^*$ with $|u| = 1$ such that $f(z+1) = uf(z)$ and the expansion of f at ∞ takes the form $\sum_{\lambda \in \Lambda} a(\lambda)e^{2\pi i \lambda z}$, where $\Lambda = \{\lambda \in \mathbf{C} \mid e^{2\pi i \lambda} = u\}$. The conclusion

that “ f is strongly non-lacunary” of theorem 2 and theorem 3 can then be strengthened as follows: there exists $\lambda_0 \in \mathbf{R}$ such that $a(\lambda) \neq 0$ for all $\lambda \geq \lambda_0$ in Λ . Moreover, such a λ_0 can be determined effectively.

Theorem 1 is not very deep and has just been stated for the sake of completeness. In fact any meromorphic function g on \mathfrak{H} satisfying the conditions stated at the beginning of the introduction, having finite order at ∞ and whose expansion at ∞ is lacunary, is necessarily holomorphic on \mathfrak{H} . This will be proved in section 5.

Theorem 2 will be deduced from the Hardy-Ramanujan circle method (see [3]), as generalized by Rademacher (see [4], Chapters 14 and 15). We shall present this theory in section 1 in the special case where $\Gamma = \mathbf{SL}_2(\mathbf{Z})$: if f is a modular function for $\mathbf{SL}_2(\mathbf{Z})$ of some real weight $k < 0$ for some automorphy factors, Rademacher’s theory yields convergent series expressing the coefficients of the expansion of f at ∞ in terms of those of its polar part (and in particular closed formulas for the partition function $p(n)$ when f is the inverse of the Dedekind eta function). A variant of this theory leading to the same results but using Poincaré series will be presented in section 2.

Rademacher’s theory can be extended to the case where $k \geq 0$ except that in this case the formulas for the coefficients involve an error term. This will be explained in section 3 and theorem 3 will follow in the case $\Gamma = \mathbf{SL}_2(\mathbf{Z})$.

In section 4, we shall explain how to extend theorems 2 and 3 to any finite index subgroup Γ of $\mathbf{SL}_2(\mathbf{Z})$.

The motivation of this paper arose from [1] and [2], where classifications of the lacunary eta-products of some particular types were given. All those found had a non-negative weight and were finite at the cusps. Our goal was to find a theoretical explanation for this observation. This explains the degree of generality (arbitrary real weight, automorphy factors, finite index subgroup of $\mathbf{SL}_2(\mathbf{Z})$) chosen to write this paper.

§1. THE CIRCLE METHOD FOR NEGATIVE WEIGHT, FOLLOWING RADEMACHER

1.0. Assumptions and notations

In this section, we consider a non-zero modular function f for $\mathbf{SL}_2(\mathbf{Z})$ of some real weight k , *holomorphic on \mathfrak{H}* . We denote by $j(\gamma, z)$ its automorphy factors. Hence we have

$$f(\gamma z) = j(\gamma, z)f(z) \quad \text{and} \quad |j(\gamma, z)| = |cz + d|^k$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ and z in \mathfrak{H} . Moreover, the functions $j(\gamma, z)$ satisfy the cocycle relation $j(\gamma\gamma', z) = j(\gamma, \gamma'z)j(\gamma', z)$. The function $j(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z)$ takes a

constant value u . We have $f(z+1) = uf(z)$ for $z \in \mathfrak{H}$ and $|u| = 1$. Let

$$f(z) = \sum_{\lambda \in \Lambda} a(\lambda) e^{2\pi i \lambda z}$$

be the expansion of f at ∞ , where Λ is the set of $\lambda \in \mathbf{R}$ such that $e^{2\pi i \lambda} = u$. By definition of a modular function there are only finitely many $\lambda < 0$ in Λ such that $a(\lambda) \neq 0$. We call

$$P(z) = \sum_{\lambda \in \Lambda, \lambda < 0} a(\lambda) e^{2\pi i \lambda z},$$

the *polar part of f at ∞* .

1.1. The circle method

For each rational number x written in an irreducible form $\frac{a}{c}$, let C_x denote the circle in the upper half plane of radius $1/2c^2$ tangent to the real line at the point x . These circles are called the *Ford circles*. We orient them in the clockwise direction and denote by C_x^* the circle C_x with the point x removed.

Proposition 1. – *Assumptions and notations are those of subsection 1.0. Further, we assume the weight k of f to be < 0 . For each $\lambda \in \Lambda$, we have*

$$(4) \quad a(\lambda) = \sum_x \int_{C_x^*} f(z) e^{-2\pi i \lambda z} dz,$$

where x runs over a system of representatives of \mathbf{Q} modulo \mathbf{Z} . Moreover, the integrals and the series in this expression converge absolutely.

We note that the function $z \mapsto f(z) e^{-2\pi i \lambda z}$ is invariant by $z \mapsto z+1$. Hence our series does not depend on the choice of the system of representatives. By (2), we know that

$$(5) \quad a(\lambda) = \int_{[i, i+1]} f(z) e^{-2\pi i \lambda z} dz.$$

Since f is holomorphic on \mathfrak{H} , we can replace the segment of integration $[i, i+1]$ by any other continuous path connecting i to $i+1$ in \mathfrak{H} .

We shall construct such a path \mathbf{c}_N for each integer $N \geq 1$. For this purpose, we consider the ordered sequence of rational numbers in $[0, 1]$ whose denominators are less than or equal to N : this sequence is called the *Farey sequence of order N* . Two consecutive terms $x' = \frac{a'}{c'}$ and $x = \frac{a}{c}$ of this sequence, where the fractions are in irreducible form with positive denominators, are such that $ac' - a'c = 1$. Hence the corresponding Ford circles $C_{x'}$ and C_x are tangent. The path \mathbf{c}_N is constructed by starting at i on the first circle C_0 and moving along it in the clockwise direction till it meets the second circle and then moving similarly along the second circle. Continuing this we reach $i+1$ through

an arc of the last Ford circle C_1 in the clockwise direction. (For more details see [4], section 116.)

When N goes to ∞ , the arc of a given Ford circle C_x contained in \mathbf{c}_N increases and its limit is C_x^* if $x \in]0, 1[$, the right half of C_0^* if $x = 0$ and the left half of C_1^* if $x = 1$. Hence proposition 1 follows from the equality $a(\lambda) = \int_{\mathbf{c}_N} f(z) e^{-2\pi i \lambda z} dz$ by passing to the limit, if we are able to prove that the sum $\sum_{x \in [0, 1[\cap \mathbf{Q}} \int_{C_x^*} |f(z) e^{-2\pi i \lambda z}| |dz|$ is finite.

Let x be a rational number. We can choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ such that $c > 0$ and $x = \frac{a}{c}$. Under the change of variable $z \mapsto \gamma z$, the line $i + \mathbf{R}$ goes to C_x^* and so we have

$$\begin{aligned}
 \int_{C_x^*} |f(z) e^{-2\pi i \lambda z}| |dz| &= \int_{i+\mathbf{R}} |f(\gamma z) e^{-2\pi i \lambda \gamma z}| |d(\gamma z)| \\
 (6) \qquad \qquad \qquad &= \int_{i+\mathbf{R}} |j(\gamma, z) f(z) e^{-2\pi i \lambda \gamma z}| |d(\gamma z)| \\
 &= \int_{i+\mathbf{R}} |cz + d|^{k-2} |f(z) e^{-2\pi i \lambda \gamma z}| |dz|.
 \end{aligned}$$

Now the function f is bounded on $i + \mathbf{R}$. We have

$$|e^{-2\pi i \lambda \gamma z}| = e^{2\pi \lambda \operatorname{Im}(\gamma z)} \leq e^{2\pi \lambda_+ / c^2} \leq e^{2\pi \lambda_+}$$

for $z \in i + \mathbf{R}$, where $\lambda_+ = \max(0, \lambda)$, and

$$\int_{i+\mathbf{R}} |cz + d|^{k-2} |dz| = c^{k-2} \int_{i+\mathbf{R}} |z|^{k-2} |dz|.$$

Therefore there exists a constant $A > 0$, independent of x , such that

$$(7) \qquad \int_{C_x^*} |f(z) e^{-2\pi i \lambda z}| |dz| \leq A c^{k-2}.$$

The sum of these integrals for $x \in [0, 1[\cap \mathbf{Q}$ is finite: indeed, for each $c \geq 1$, the number of rational numbers in $[0, 1[$ with denominator c is $\varphi(c)$ (where φ denotes the Euler function), hence is at most c ; and since $k < 0$, the series $\sum_{c=1}^{\infty} c^{k-1}$ converges. This completes the proof.

Remark. — The previous proof shows that if we assume $k < 1$ in place of $k < 0$, each integral $\int_{C_x^*} f(z) e^{-2\pi i \lambda z} dz$ still makes sense and is absolutely convergent.

1.2. Some special functions

To compute the integrals occurring in proposition 1, we shall need to study the functions L_ν defined for ν and t in \mathbf{C} by

$$(8) \qquad L_\nu(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! \Gamma(n + \nu + 1)},$$

where Γ is the Gamma function. Since $1/\Gamma$ is an entire function and since the series (8) converges normally on any compact set of \mathbf{C}^2 , the map $(\nu, t) \mapsto L_\nu(t)$ is holomorphic on \mathbf{C}^2 . The function L_ν is essentially a Bessel function: more precisely, the modified Bessel function of the first kind, classically denoted by $I_\nu(t)$, is equal to $(\frac{t}{2})^\nu L_\nu(\frac{t^2}{4})$.

Proposition 2.— *For $\nu \in \mathbf{C}$ with $\operatorname{Re}(\nu) > 0$ and $t \in \mathbf{C}$, we have*

$$(9) \quad L_\nu(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} w^{-\nu-1} e^{w+\frac{t}{w}} dw,$$

where σ is any positive real number and the complex power $w^{-1-\nu}$ is defined as in the introduction.

Before proving proposition 2, we shall recall two versions of Hankel's formula for the inverse of the Γ function.

Lemma 1.— *a) Let σ be a positive real number. We have*

$$(10) \quad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^w w^{-s} dw$$

for all $s \in \mathbf{C}$ with $\operatorname{Re}(s) > 1$.

b) Let σ and β be positive real numbers. Let $H_{\sigma,\beta}$ denote the following Hankel contour: it consists of the horizontal half line from $-\infty - i\beta$ to $\sigma - i\beta$, followed by the vertical segment from $\sigma - i\beta$ to $\sigma + i\beta$ and by the horizontal half line from $\sigma + i\beta$ to $-\infty + i\beta$. We have

$$(11) \quad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{H_{\sigma,\beta}} e^w w^{-s} dw$$

for all $s \in \mathbf{C}$.

For $w \in \mathbf{C} - \mathbf{R}_-$, we have

$$(12) \quad |w^{-s}| = e^{-\operatorname{Re}(s) \log |w|} \leq e^{-\operatorname{Re}(s) \log |w| + \pi |\operatorname{Im}(s)|} \leq |w|^{-\operatorname{Re}(s)} e^{\pi |\operatorname{Im}(s)|}$$

and $|e^w| = e^{\operatorname{Re}(w)}$. It follows that the integral in (10) converges for $\operatorname{Re}(s) > 1$, that the integral in (11) converges for all $s \in \mathbf{C}$ and that these two integrals are equal when $\operatorname{Re}(s) > 1$. It therefore suffices to prove assertion b). Since both sides of (11) are entire functions of s , it even suffices to prove (11) when $\operatorname{Re}(s) < 0$. In this case, since the right hand side of (11) does not depend on β , we can replace it by its limit when β goes to 0. This limit is

$$\frac{1}{2\pi i} \int_{-\infty}^0 e^w |w|^{-s} (e^{s\pi i} - e^{-s\pi i}) dw = \frac{\sin \pi s}{\pi} \int_0^\infty e^{-w} w^{-s} dw = \frac{\sin \pi s}{\pi} \Gamma(1-s).$$

It is equal to $\frac{1}{\Gamma(s)}$ by the reflection formula. This completes the proof of the lemma.

We now prove proposition 2. Applying formula (10) for $s = n + \nu + 1$, we get from (8)

$$L_\nu(t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\sigma-i\infty}^{\sigma+i\infty} e^w w^{-n-\nu-1} dw,$$

for $\nu \in \mathbf{C}$ such that $\operatorname{Re}(\nu) > 0$ and $t \in \mathbf{C}$. Proposition 2 follows by interchanging the sum and the integral. In order to justify this interchange, we shall prove that the integral

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n=0}^{\infty} \frac{|t|^n}{n!} |e^w w^{-n-\nu-1}| |dw| = \int_{\sigma-i\infty}^{\sigma+i\infty} e^{|t|/|w|} |e^w w^{-\nu-1}| |dw|$$

is finite. The function $w \mapsto e^{|t|/|w|} |e^w|$ is bounded on the line of integration. By (12), we have $|w^{-\nu-1}| = O(|w|^{-\operatorname{Re}(\nu)-1})$ when w goes to infinity on this line. Since $\operatorname{Re}(\nu) > 0$, $\int_{\sigma-i\infty}^{\sigma+i\infty} |w|^{-\operatorname{Re}(\nu)-1} |dw|$ converges. This completes the proof.

Remark. – Let $H_{\sigma,\beta}$ be the Hankel contour defined in lemma 1, b). We have

$$(13) \quad L_\nu(t) = \frac{1}{2\pi i} \int_{H_{\sigma,\beta}} w^{-\nu-1} e^{w+\frac{t}{w}} dw$$

for all $\nu \in \mathbf{C}$. The proof is similar to the proof of proposition 2 except that we use formula (11) instead of formula (10) and the fact that $\int_{H_{\sigma,\beta}} |w^{-\nu-1} e^w| |dw|$ converges.

1.3. Computation of integrals

In this subsection, we compute the integrals occurring in proposition 1. We adopt the hypotheses and notations of subsection 1.0, and we moreover assume the weight k of f to be < 0 .

Proposition 3. – *Let λ be any element in Λ , x be a rational number and C_x denote the corresponding Ford circle. Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ with $c > 0$ such that $x = \frac{a}{c}$. Write $j(\gamma, z) = \varepsilon(\gamma) \left(\frac{cz+d}{i} \right)^k$. The integral $\int_{C_x^*} f(z) e^{-2\pi i \lambda z} dz$ is equal to*

$$(14) \quad (2\pi)^{2-k} c^{k-2} \sum_{\mu \in \Lambda, \mu < 0} a(\mu) |\mu|^{1-k} \varepsilon(\gamma) e^{-2\pi i \frac{\lambda a + \mu d}{c}} L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}).$$

Under the change of variable $z \mapsto \gamma z$, the line $i + \mathbf{R}$ goes to C_x^* . The negative orientation of $i + \mathbf{R}$ corresponds to the clockwise orientation of C_x^* . Hence we have

$$(15) \quad \int_{C_x^*} f(z) e^{-2\pi i \lambda z} dz = \int_{i+\infty}^{i-\infty} j(\gamma, z) f(z) e^{-2\pi i \lambda \gamma z} d(\gamma z).$$

Lemma 2.— *We have*

$$(16) \quad \int_{i+\infty}^{i-\infty} j(\gamma, z)(f(z) - P(z))e^{-2\pi i \lambda \gamma z} d(\gamma z) = 0.$$

The function $f - P$ is bounded on the half plane $\overline{\mathfrak{H}}_1 = \{z \in \mathbf{C} | \text{Im}(z) \geq 1\}$. The same is true of the function $z \mapsto e^{-2\pi i \lambda \gamma z}$ since

$$|e^{-2\pi i \lambda \gamma z}| = e^{2\pi \lambda \text{Im}(z)/|cz+d|^2} \leq e^{2\pi \lambda_+/c^2}$$

for $z \in \overline{\mathfrak{H}}_1$, where $\lambda_+ = \max(0, \lambda)$. We have $|j(\gamma, z)| \left| \frac{d(\gamma z)}{dz} \right| = |cz + d|^{k-2}$ and k is negative, hence the left hand side of (16) remains unchanged when we replace $\int_{i+\infty}^{i-\infty}$ by $\int_{it+\infty}^{it-\infty}$ for any $t \geq 1$, and its absolute value is bounded by a constant times

$$\int_{it+\mathbf{R}} |cz + d|^{k-2} |dz| = c^{k-2} t^{k-1} \int_{i+\mathbf{R}} |z|^{k-2} |dz|.$$

Lemma 2 follows by letting t go to ∞ .

From (15) and (16), we deduce

$$(17) \quad \begin{aligned} \int_{C_x^*} f(z) e^{-2\pi i \lambda z} dz &= \int_{i+\infty}^{i-\infty} j(\gamma, z) P(z) e^{-2\pi i \lambda \gamma z} d(\gamma z) \\ &= \sum_{\mu \in \Lambda, \mu < 0} a(\mu) \varepsilon(\gamma) \int_{i-\infty}^{i+\infty} \left(\frac{cz + d}{i} \right)^{k-2} e^{2\pi i (\mu z - \lambda \frac{az+b}{cz+d})} dz \end{aligned}$$

By making the change of variable $w = 2\pi i \mu (z + \frac{d}{c})$, we see that the last expression is equal to

$$\sum_{\mu \in \Lambda, \mu < 0} a(\mu) \varepsilon(\gamma) e^{-2\pi i \frac{\lambda a + \mu d}{c}} \left(\frac{c}{2\pi |\mu|} \right)^{k-2} \int_{2\pi |\mu| - i\infty}^{2\pi |\mu| + i\infty} w^{k-2} e^{w + \frac{4\pi^2 \lambda |\mu|}{c^2 w}} \frac{dw}{2\pi i |\mu|}$$

and by proposition 2, the last integral is equal to $|\mu|^{-1} L_{1-k}(4\pi^2 \lambda |\mu| c^{-2})$. This proves proposition 3.

Remark 1. — Let $\lambda, \mu \in \Lambda$. The complex number $\varepsilon(\gamma) e^{-2\pi i \frac{\lambda a + \mu d}{c}}$ has absolute value 1 and depends only on λ, μ and x and not on the choice of the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ such that $c > 0$ and $x = \frac{a}{c}$. Moreover, it remains unchanged if we translate x by an integer. These statements are proved by checking that if we replace γ by $\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \gamma$, $\varepsilon(\gamma)$ gets multiplied by u^n and $e^{-2\pi i \frac{\lambda a + \mu d}{c}}$ by u^{-n} , where u is the constant value taken by $j(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z)$. It will be convenient to adopt the notation $\alpha_x(\lambda, \mu)$ for the complex number $\varepsilon(\gamma) e^{-2\pi i \frac{\lambda a + \mu d}{c}}$.

Remark 2. — As we have seen in the remark of subsection 1.1, the integrals $\int_{C_x^*} f(z) e^{-2\pi i \lambda z} dz$ still converge if we assume $k < 1$ in place of $k < 0$. Proposition 3 remains valid in this context with the same proof.

1.4. Rademacher's formula for the coefficients of f

Hypotheses and notations are as in 1.0 and we moreover assume the weight k of f to be < 0 . For any pair (λ, μ) of elements of Λ and any integer $c \geq 1$, let $A_c(\lambda, \mu)$ denote the sum of the complex numbers $\alpha_x(\lambda, \mu)$ defined in the remark 1 of 1.3, where $x = \frac{a}{c}$ and a runs over the integers in $[0, c[$ coprime to c . Since this sum has exactly $\varphi(c)$ terms and since $|\alpha_x(\lambda, \mu)| = 1$ for each of them, we have $|A_c(\lambda, \mu)| \leq \varphi(c) \leq c$.

By combining proposition 1 and proposition 3, we get Rademacher's formula expressing all coefficients of the expansion of f at ∞ in terms of those of its polar part.

Theorem 4.— *Let $\lambda \in \Lambda$. With the previous notations, we have*

$$(18) \quad a(\lambda) = (2\pi)^{2-k} \sum_{c=1}^{\infty} c^{k-2} \sum_{\mu \in \Lambda, \mu < 0} a(\mu) |\mu|^{1-k} A_c(\lambda, \mu) L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}).$$

Remark 1.— Theorem 4 applied to η^{-1} , where η is the Dedekind eta function yields Rademacher's formula for the number of partitions $p(n)$ of an integer n . Note that in this case k is equal to $-\frac{1}{2}$ and $L_{\frac{3}{2}}(t)$ to $\frac{1}{\sqrt{\pi}} \frac{d}{dt} \left(\frac{\text{sh}(2\sqrt{t})}{\sqrt{t}} \right)$.

Remark 2.— The automorphy factor $j\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z\right)$ can be written as $\varepsilon \left(\frac{z}{i}\right)^k$ for some $\varepsilon \in \mathbf{C}$, and we have $j\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, z\right) = 1$. By applying the cocycle relation (3) to $\gamma = \gamma' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $z = i$, we get $\varepsilon^2 = 1$. Hence ε is equal to 1 or -1 . We note that $A_1(\lambda, \mu) = \varepsilon$ for any pair $(\lambda, \mu) \in \Lambda^2$, as one sees by computing $\alpha_0(\lambda, \mu)$ using the matrix $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

1.5. Proof of theorem 2 when $\Gamma = \text{SL}_2(\mathbf{Z})$

Assumptions and notations are as in subsection 1.0 and we moreover assume the weight k of f to be < 0 . There exists at least one $\mu < 0$ in Λ such that $a(\mu) \neq 0$, otherwise f would be equal to 0 by theorem 4. Let μ_0 be the one with largest absolute value. Let $\delta = |\mu_0|$ and δ' denote the second largest of the numbers $|\mu|c^{-2}$, where μ is a negative element in Λ such that $a(\mu) \neq 0$ and c is a positive integer.

Proposition 4.— *For each $\lambda \geq 0$ in Λ , we have*

$$(19) \quad |a(\lambda) - (2\pi)^{2-k} a(\mu_0) \delta^{1-k} \varepsilon L_{1-k}(4\pi^2 \lambda \delta)|$$

$$(20) \quad \leq (2\pi)^{2-k} \zeta(1-k) M(f) \delta^{1-k} L_{1-k}(4\pi^2 \lambda \delta'),$$

where ζ denotes the Riemann zeta function, $M(f)$ is equal to $\sum_{\mu \in \Lambda, \mu < 0} |a(\mu)|$ and $\varepsilon = \pm 1$ is defined in remark 2 of subsection 1.4.

We apply theorem 4. By remark 2 of subsection 1.4, the term in the double sum (18) corresponding to $c = 1$ and $\mu = \mu_0$ is $(2\pi)^{2-k} a(\mu_0) \delta^{1-k} \varepsilon L_{1-k}(4\pi^2 \lambda \delta)$. We bound the absolute value of all the terms corresponding to the other pairs (c, μ) such that $a(\mu) \neq 0$ by using the following facts:

a) By formula (8), the function L_{1-k} is non-negative and increasing on \mathbf{R}_+ . Hence $0 \leq L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}) \leq L_{1-k}(4\pi^2 \lambda \delta')$.

b) We have $|\mu|^{1-k} \leq \delta^{1-k}$, $|A_c(\lambda, \mu)| \leq c$ and $\sum_{c=1}^{\infty} c^{k-1} = \zeta(1-k)$.

Corollary 1.— *The coefficient $a(\lambda)$ is equivalent to*

$$(21) \quad (2\pi)^{2-k} a(\mu_0) \delta^{1-k} \varepsilon L_{1-k}(4\pi^2 \lambda \delta)$$

when $\lambda \in \Lambda$ goes to $+\infty$.

Since $\varepsilon = \pm 1$ and $a(\mu_0) \neq 0$, the corollary follows from proposition 4 if we prove that $L_{1-k}(4\pi^2 \lambda \delta') / L_{1-k}(4\pi^2 \lambda \delta)$ goes to 0 when $\lambda \rightarrow +\infty$. This can be deduced from the classical fact in the theory of Bessel functions that $L_{1-k}(t)$ is equivalent to $\frac{1}{\sqrt{4\pi}} t^{\frac{2k-3}{4}} e^{2\sqrt{t}}$ when $t \rightarrow +\infty$, but it is also a consequence of the following more elementary lemma, applied to the function L_{1-k} .

Lemma 3.— *Let $L \neq 0$ be an entire function defined by a formal power series with non-negative coefficients $\sum_{n=0}^{\infty} b_n t^n$. Let α' and α be two positive real numbers such that $\alpha' < \alpha$. The function $t \mapsto \frac{L(t\alpha')}{L(t\alpha)}$ is decreasing on \mathbf{R}_+^* . Its limit when $t \rightarrow +\infty$ is 0 if L is not a polynomial.*

Let t' and t be two positive real numbers such that $t' \leq t$. We have

$$\begin{aligned} L(t\alpha)L(t'\alpha') - L(t\alpha')L(t'\alpha) &= \sum_{n,m} b_n b_m (t^n \alpha^n t'^m \alpha'^m - t^n \alpha'^n t'^m \alpha^m) \\ &= \sum_{n,m} b_n b_m t^n t'^m (\alpha^n \alpha'^m - \alpha'^n \alpha^m) \\ &= \sum_{n>m} b_n b_m (t^n t'^m - t^m t'^n) (\alpha^n \alpha'^m - \alpha'^n \alpha^m) \geq 0, \end{aligned}$$

hence $\frac{L(t'\alpha')}{L(t'\alpha)} \geq \frac{L(t\alpha')}{L(t\alpha)}$. This proves that the function $t \mapsto \frac{L(t\alpha')}{L(t\alpha)}$ is decreasing on \mathbf{R}_+^* . Let ℓ denote its limit when $t \rightarrow +\infty$. If L is not a polynomial, then for each $N \in \mathbf{N}$, $L(t)$ is equivalent to $L_N(t) = \sum_{n=N}^{\infty} b_n t^n$ when $t \rightarrow +\infty$. But $L_N(t\alpha') \leq \left(\frac{\alpha'}{\alpha}\right)^N L_N(t\alpha)$ for all $t \in \mathbf{R}_+$, hence $\ell \leq \left(\frac{\alpha'}{\alpha}\right)^N$. We thus have $\ell = 0$.

Corollary 2.— *There exists $\lambda_0 \in \mathbf{R}$ such that $a(\lambda) \neq 0$ for all $\lambda \geq \lambda_0$ in Λ .*

Such a real number λ_0 can be determined effectively: indeed by proposition 4 and lemma 3, it suffices to choose λ_0 in \mathbf{R}_+ such that

$$(22) \quad \frac{L_{1-k}(4\pi^2\lambda_0\delta')}{L_{1-k}(4\pi^2\lambda_0\delta)} < \zeta(1-k)^{-1} \frac{|a(\mu_0)|}{\sum_{\mu \in \Lambda, \mu < 0} |a(\mu)|}.$$

§2. A VARIANT OF THE CIRCLE METHOD

In this section, we shall present a variant of Rademacher's circle method based on the theory of Poincaré series. Assumptions and notations are the same as in subsection 1.0. Moreover, we suppose that the weight k of f is < 0 . Recall that the automorphy factors of f are denoted by $j(\gamma, z)$ for $\gamma \in \mathbf{SL}_2(\mathbf{Z})$, that u denotes the constant value of $j\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z\right)$ and $\sum_{\lambda \in \Lambda} a(\lambda)e^{2\pi i\lambda z}$ the expansion of f at ∞ , where $\Lambda = \{\lambda \in \mathbf{C} | e^{2\pi i\lambda} = u\}$.

2.1. Principle of the method

In subsection 2.2, we shall construct for each λ in Λ a Poincaré series E_λ with the following properties:

a) It is a modular function of weight $2 - k$ for $\mathbf{SL}_2(\mathbf{Z})$, holomorphic on \mathfrak{H} , with automorphy factors $j(\gamma, z)^{-1}(cz + d)^2$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$. We then have $E_\lambda(z + 1) = u^{-1}E_\lambda(z)$.

b) The function $E_\lambda(z) - e^{-2\pi i\lambda z}$ goes to 0 when $\text{Im}(z)$ goes to $+\infty$. It follows from a) and b) that $E_\lambda(z) - e^{-2\pi i\lambda z}$ has an expansion at ∞ of the form

$$(23) \quad E_\lambda(z) - e^{-2\pi i\lambda z} = \sum_{\mu \in \Lambda, \mu < 0} e(\lambda, \mu)e^{-2\pi i\mu z}.$$

The coefficients $e(\lambda, \mu)$ will be computed in subsection 2.3.

The differential form $f(z)E_\lambda(z)dz$ is then holomorphic on \mathfrak{H} and invariant under the action of $\mathbf{SL}_2(\mathbf{Z})$. It defines by passing to the quotient a holomorphic differential form on the Riemann surface $\mathbf{SL}_2(\mathbf{Z}) \backslash \mathfrak{H}$. The modular invariant identifies this Riemann surface with $\mathbf{P}_1(\mathbf{C}) - \{\infty\}$. But any holomorphic differential form on $\mathbf{P}_1(\mathbf{C}) - \{\infty\}$ has residue 0 at ∞ . Therefore we have

$$\int_{[i, i+1]} f(z)E_\lambda(z)dz = 0.$$

We use the expansions of f and E_λ at ∞ to compute this integral and we get

$$(24) \quad a(\lambda) + \sum_{\mu \in \Lambda, \mu < 0} a(\mu)e(\lambda, \mu) = 0.$$

Replacing the coefficients $e(\lambda, \mu)$ by their expressions obtained in subsection 2.3, we shall obtain the same formulas as in theorem 4 for λ in Λ .

2.2. The Poincaré series E_λ

Let U denote the stabilizer of ∞ in $\mathbf{SL}_2(\mathbf{Z})$: it is generated by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We have $j(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, z) = 1$ and $j(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z) = u$.

Let $\lambda \in \Lambda$. Since $e^{2\pi i \lambda} = u$, the differential form $j(\gamma, z)e^{-2\pi i \lambda \gamma z} d(\gamma z)$ on \mathfrak{H} depends only on the coset $U\gamma$ in $U \backslash \mathbf{SL}_2(\mathbf{Z})$. We define a function E_λ on \mathfrak{H} by

$$(25) \quad E_\lambda(z) dz = \sum_{\gamma \in U \backslash \mathbf{SL}_2(\mathbf{Z})} j(\gamma, z) e^{-2\pi i \lambda \gamma z} d(\gamma z).$$

The function E_λ is the sum of $e^{-2\pi i \lambda z}$ and of the series

$$(26) \quad \sum_{\gamma \in U \backslash (\mathbf{SL}_2(\mathbf{Z}) - U)} j(\gamma, z) e^{-2\pi i \lambda \gamma z} \frac{d(\gamma z)}{dz}.$$

Lemma 4.— *The series (26) converges normally on any half plane \mathfrak{H}_r with $r > 0$ and its sum goes to 0 when $\text{Im}(z)$ goes to $+\infty$.*

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}) - U$. Then $c \neq 0$ and we have $|j(\gamma, z) \frac{d(\gamma z)}{dz}| = |cz + d|^{k-2}$ and $|e^{-2\pi i \lambda \gamma z}| = e^{2\pi \lambda \text{Im}(z)/|cz+d|^2} \leq e^{2\pi \lambda_+ / \text{Im}(z)} \leq e^{2\pi \lambda_+ / r}$ on \mathfrak{H}_r .

The map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \text{sign}(c)(c, d)$ defines by passing to the quotient a bijection from $U \backslash (\mathbf{SL}_2(\mathbf{Z}) - U)$ onto the set of pairs of coprime integers (c, d) such that $c \geq 1$. The sum over these pairs of $|cz + d|^{k-2}$ is finite because $k < 0$. This proves that the series (26) converges normally on \mathfrak{H}_r . Since each of its terms goes to 0 when $\text{Im}(z)$ goes to $+\infty$, the same is true of its sum.

It follows from lemma 4 that E_λ is a holomorphic function on \mathfrak{H} . It follows from formula (25) and from the cocycle relation (3) that the holomorphic differential form $\omega_\lambda = E_\lambda(z) dz$ on \mathfrak{H} satisfies the functional equation $\gamma^*(\omega_\lambda) = j(\gamma, z)^{-1} \omega_\lambda$ for γ in $\mathbf{SL}_2(\mathbf{Z})$. Equivalently, we have $E_\lambda(\gamma z) = j(\gamma, z)^{-1} (cz + d)^2 E_\lambda(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ and $z \in \mathfrak{H}$. By lemma 4, $E_\lambda(z) - e^{-2\pi i \lambda z}$ goes to 0 when $\text{Im}(z)$ goes to ∞ . Furthermore it gets multiplied by u^{-1} when z is replaced by $z + 1$. Hence its expansion at ∞ is of the form $\sum_{\mu \in \Lambda, \mu < 0} e(\lambda, \mu) e^{-2\pi i \mu z}$.

2.3. Computation of the coefficients $e(\lambda, \mu)$

For all $\mu < 0$ in Λ , we have by lemma 4

$$(27) \quad \begin{aligned} e(\lambda, \mu) &= \int_{[i, i+1]} e^{2\pi i \mu z} (E_\lambda(z) - e^{-2\pi i \lambda z}) dz \\ &= \sum_{\gamma \in U \backslash (\mathbf{SL}_2(\mathbf{Z}) - U)} \int_{[i, i+1]} j(\gamma, z) e^{2\pi i (\mu z - \lambda \gamma z)} d(\gamma z). \end{aligned}$$

Let γ be an element of $\mathbf{SL}_2(\mathbf{Z}) - U$ and let γ_0 denote the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $U \cap \gamma U \gamma^{-1} = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \}$. Therefore the cosets $U \gamma \gamma_0^n$, for $n \in \mathbf{Z}$, are all

distinct. Their union is the double coset $U\gamma U$. Moreover, since $e^{2\pi i\mu} = u$ and $j(\gamma\gamma_0^n, z) = j(\gamma, z+n)u^n$, we have

$$\begin{aligned} & \int_{[i, i+1]} j(\gamma\gamma_0^n, z) e^{2\pi i(\mu z - \lambda\gamma\gamma_0^n z)} d(\gamma\gamma_0^n z) \\ &= \int_{[i, i+1]} j(\gamma, z+n) e^{2\pi i(\mu(z+n) - \lambda\gamma(z+n))} d(\gamma(z+n)) \\ &= \int_{[i+n, i+n+1]} j(\gamma, z) e^{2\pi i(\mu z - \lambda\gamma z)} d(\gamma z). \end{aligned}$$

Hence by grouping in (27) the terms by double cosets, we get

$$e(\lambda, \mu) = \sum_{\gamma \in U \backslash (\mathbf{SL}_2(\mathbf{Z}) - U) / U} \int_{i-\infty}^{i+\infty} j(\gamma, z) e^{2\pi i(\mu z - \lambda\gamma z)} d(\gamma z).$$

These integrals are exactly those that we already met in formula (17). Moreover, the map $\gamma \mapsto \gamma\infty$ defines by passing to the quotient a bijection from $(\mathbf{SL}_2(\mathbf{Z}) - U) / U$ to \mathbf{Q} and hence from $U \backslash (\mathbf{SL}_2(\mathbf{Z}) - U) / U$ to \mathbf{Q}/\mathbf{Z} . The same computation as in subsection 1.3 and 1.4 shows that

$$e(\lambda, \mu) = - \sum_{c=1}^{\infty} (2\pi)^{2-k} c^{k-2} |\mu|^{1-k} A_c(\lambda, \mu) L_{1-k} (4\pi^2 \lambda |\mu| c^{-2}),$$

the notations being those of subsection 1.4. This shows formula (24) is the same as theorem 4.

§3. EXTENSION OF THE CIRCLE METHOD TO NON-NEGATIVE WEIGHT

3.0. Assumptions and notations

Assumptions and notations in this section are those of subsection 1.0 but we now restrict ourselves to the case where *the weight k of the modular function f is non-negative*. We recall that the automorphy factors of f are denoted by $j(\gamma, z)$ for $\gamma \in \mathbf{SL}_2(\mathbf{Z})$, that u denotes the constant value of $j(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z)$, $\sum_{\lambda \in \Lambda} a(\lambda) e^{2\pi i \lambda z}$ the expansion of f at ∞ , where $\Lambda = \{\lambda \in \mathbf{C} | e^{2\pi i \lambda} = u\}$, and P denotes its polar part.

3.1. The circle method revisited

Let $\lambda \in \Lambda$. By (2), we know that

$$(28) \quad a(\lambda) = \int_{[i, i+1]} f(z) e^{-2\pi i \lambda z} dz.$$

Since f is holomorphic on \mathfrak{H} , we can replace the segment of integration $[i, i+1]$ by any other continuous path connecting i to $i+1$ in \mathfrak{H} , in particular by the paths \mathbf{c}_N defined in 1.1. But the procedure of letting N go to ∞ is no more justified.

Hence from now on, we fix an integer $N \geq 1$. Let \mathbf{Q}_N denote the set of rational numbers with denominator $\leq N$. It is a closed discrete subset of \mathbf{R} . Let $x \in \mathbf{Q}_N$. Let x' denote the predecessor and x'' the successor of x . The Ford circle C_x is tangent to $C_{x'}$ at a point $R_{x,N}$ and to $C_{x''}$ at a point $S_{x,N}$. Let $C_{x,N}$ denote the arc of C_x joining $R_{x,N}$ to $S_{x,N}$ in the clockwise direction. The path \mathbf{c}_N is composed of the right half of $C_{0,N}$, of the arcs $C_{x,N}$ for $x \in]0, 1[$ and of the left half of $C_{1,N}$. Hence we have

$$(29) \quad a(\lambda) = \sum_x \int_{C_{x,N}} f(z) e^{-2\pi i \lambda z} dz,$$

where x runs over a system of representatives of \mathbf{Q}_N modulo \mathbf{Z} .

3.2. Parametrization of the paths $C_{x,N}$

Let x be an element of \mathbf{Q}_N . Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ with $c > 0$ such that $x = \frac{a}{c}$. Let n and m denote the largest integers in \mathbf{Z} such that $d + nc \leq N$ and $-d + mc \leq N$ respectively. We set

$$a' = b + na, \quad c' = d + nc, \quad a'' = -b + ma, \quad c'' = -d + mc.$$

We have $1 \leq c' \leq N$, $1 \leq c'' \leq N$, $c + c' \geq N + 1$, $c + c'' \geq N + 1$. We hence have $c^2 + c'^2 \geq \frac{(c+c')^2}{2} \geq \frac{(N+1)^2}{2}$ and similarly $c^2 + c''^2 \geq \frac{(N+1)^2}{2}$. With these notations:

Proposition 5.— a) The predecessor of x in \mathbf{Q}_N is $x' = \frac{a'}{c'}$ and its successor is $x'' = \frac{a''}{c''}$.

b) We have $R_{x,N} = \frac{ac+a'c'+i}{c^2+c'^2} = \frac{a'+ia}{c'+ic}$ and $S_{x,N} = \frac{ac+a''c''+i}{c^2+c''^2} = \frac{a''-ia}{c''-ic}$.

c) The imaginary parts of $R_{x,N}$ and $S_{x,N}$ lie in the interval $[\frac{1}{2N^2}, \frac{2}{(N+1)^2}]$.

d) The oriented path $C_{x,N}$ is the image under the map $z \mapsto \gamma z$ of the oriented segment $[n+i, -m+i]$ of the line $i + \mathbf{R}$.

The point $x' = \frac{a'}{c'}$ belongs to \mathbf{Q}_N and is smaller than x since $ac' - a'c = ad - bc = 1$. Let $\frac{p}{q}$ with $q > 0$ and $\gcd(p, q) = 1$ be a rational number such that $x' < \frac{p}{q} < x$. We have $aq - cp \geq 1$ and $c'p - a'q \geq 1$, hence $q = c'(aq - cp) + c(c'p - a'q) \geq c' + c > N$. Therefore $\frac{p}{q}$ does not belong to \mathbf{Q}_N . This proves that x' is the predecessor of x in \mathbf{Q}_N . The proof that x'' is its successor is similar.

Since the radius of the circle $C_{x'}$ is $\frac{1}{2c'^2}$ and the radius of C_x is $\frac{1}{2c^2}$, the point $R_{x,N}$ is the barycenter of the centers $\frac{a'}{c'} + \frac{i}{2c'^2}$ and $\frac{a}{c} + \frac{i}{2c^2}$ of these circles with coefficients $\frac{1}{2c^2}$ and $\frac{1}{2c'^2}$, or equivalently c'^2 and c^2 . We hence have $R_{x,N} = \frac{ac+a'c'+i}{c^2+c'^2} = \frac{a'+ia}{c'+ic}$. Since $\frac{(N+1)^2}{2} \leq c^2 + c'^2 \leq 2N^2$, $\text{Im}(R_{x,N}) = \frac{1}{c^2+c'^2}$ belongs to $[\frac{1}{2N^2}, \frac{2}{(N+1)^2}]$. This proves b) and c) for $R_{x,N}$. The proofs for $S_{x,N}$ are similar.

The map $z \mapsto \gamma z$ defines a diffeomorphism from $i + \mathbf{R}$ to $C_x - \{x\}$, the negative orientation of $i + \mathbf{R}$ corresponding to the clockwise orientation of C_x . This map sends $n + i$ to $R_{x,N}$ and $-m + i$ to $S_{x,N}$ by b). This proves d).

Corollary.— *a) Both points $R_{x,N}$ and $S_{x,N}$ are at a distance $\leq \frac{\sqrt{2}}{cN}$ from x .*

b) The image of the chord $[R_{x,N}, S_{x,N}]$ of C_x by γ^{-1} is an arc of a circle of radius $\leq \frac{N^2}{c^2}$.

We have $R_{x,N} - x = \frac{a' + ia}{c' + ic} - \frac{a}{c} = \frac{-1}{c(c' + ic)}$. The distance between $R_{x,N}$ and x is therefore $\frac{1}{c\sqrt{c^2 + c'^2}}$. Since $c^2 + c'^2 \geq \frac{N^2}{2}$, this distance is $\leq \frac{\sqrt{2}}{cN}$. This proves a) for $R_{x,N}$. The proof for $S_{x,N}$ is similar.

Let D denote the line passing through the points $R_{x,N}$ and $S_{x,N}$ and δ denote its distance to x . Since $R_{x,N}$ lies on the left half and $S_{x,N}$ on the right half of the circle C_x , the point of D closest to x belongs to the chord $[R_{x,N}, S_{x,N}]$. The ordinate of this point is $\geq \frac{1}{2N^2}$ by proposition 5, c), hence $\delta \geq \frac{1}{2N^2}$. The map $z \mapsto \gamma^{-1}z = \frac{dz-b}{-cz+a} = -\frac{d}{c} - \frac{1}{c^2(z-\frac{a}{c})}$ is obtained by composing the translation $z \mapsto z - \frac{a}{c}$, the transformation $z \mapsto -\frac{1}{c^2z}$ and the translation $z \mapsto z - \frac{d}{c}$. The first one maps D to a line D' whose distance to the origin is δ , the second one maps D' into a circle C' of radius $\frac{1}{2c^2\delta}$ and the third one maps C' onto a circle of the same radius. Since $\frac{1}{2c^2\delta} \leq \frac{N^2}{c^2}$, assertion b) follows.

3.3. First error term: comparing f to its polar part

We keep the notations of subsection 3.2. It follows from proposition 5, d) that we have

$$(30) \quad \int_{C_{x,N}} f(z) e^{-2\pi i \lambda z} dz = \int_{n+i}^{-m+i} j(\gamma, z) f(z) e^{-2\pi i \lambda \gamma z} d(\gamma z)$$

for all $\lambda \in \Lambda$.

We recall that P denotes the polar part $\sum_{\mu \in \Lambda, \mu < 0} a(\mu) e^{2\pi i \mu z}$ of f at ∞ . The function $|f - P|$ is bounded on $\overline{\mathfrak{H}}_1 = \{z \in \mathbf{C} \mid \text{Im}(z) \geq 1\}$. We denote by $\|f - P\|_{\overline{\mathfrak{H}}_1}$ its supremum on $\overline{\mathfrak{H}}_1$.

Proposition 6.— *For each $\lambda \in \Lambda$, $|\int_{n+i}^{-m+i} j(\gamma, z)(f(z) - P(z))e^{-2\pi i \lambda \gamma z} d(\gamma z)|$ is bounded by*

$$(31) \quad \pi e^{4\pi \lambda_+ / (N+1)^2} \|f - P\|_{\overline{\mathfrak{H}}_1} \times \begin{cases} (2N^2)^{k-1} c^{-k} & \text{if } k \geq 2, \\ 2N^{\frac{3k}{2}-1} c^{-\frac{k}{2}-1} & \text{if } 0 \leq k \leq 2. \end{cases}$$

The integral $I = \int_{n+i}^{-m+i} j(\gamma, z)(f(z) - P(z))e^{-2\pi i \lambda \gamma z} d(\gamma z)$ is unchanged if we replace the path of integration $[n+i, -m+i]$ by any other continuous path from $n+i$ to $-m+i$, for example by $\gamma^{-1}([R_{x,N}, S_{x,N}])$. Since $[R_{x,N}, S_{x,N}]$ is contained in the closed disc bounded by C_x , the path $\gamma^{-1}([R_{x,N}, S_{x,N}])$ is contained in

the closed half plane $\overline{\mathfrak{H}}_1$ bounded by $i + \mathbf{R}$. Hence $|f - P|$ is bounded by $\|f - P\|_{\overline{\mathfrak{H}}_1}$ on this path. By proposition 5, c), $|e^{-2\pi i \lambda \gamma z}| = e^{2\pi \lambda \text{Im}(\gamma z)}$ is bounded by $e^{4\pi \lambda_+ / (N+1)^2}$ on this path. We thus have

$$|I| \leq e^{4\pi \lambda_+ / (N+1)^2} \|f - P\|_{\overline{\mathfrak{H}}_1} \int_{\gamma^{-1}([R_{x,N}, S_{x,N}])} |cz + d|^{k-2} |dz|.$$

We shall now distinguish two cases:

a) We have $k \geq 2$. We have $|cz + d| = |-c(\gamma z) + a|^{-1} \leq (c \text{Im}(\gamma z))^{-1} \leq 2N^2 c^{-1}$ by proposition 5, c). Furthermore the length of the path $\gamma^{-1}([R_{x,N}, S_{x,N}])$ is at most $2\pi \frac{N^2}{c^2}$ by corollary to proposition 5. Hence we have

$$\int_{\gamma^{-1}([R_{x,N}, S_{x,N}])} |cz + d|^{k-2} |dz| \leq \pi (2N^2)^{k-1} c^{-k}.$$

b) We have $0 \leq k \leq 2$. By Hölder's inequality, $\int_{\gamma^{-1}([R_{x,N}, S_{x,N}])} |cz + d|^{k-2} |dz|$ is bounded by

$$\left(\int_{\gamma^{-1}([R_{x,N}, S_{x,N}])} |cz + d|^{-2} |dz| \right)^{1-\frac{k}{2}} \left(\int_{\gamma^{-1}([R_{x,N}, S_{x,N}])} |dz| \right)^{\frac{k}{2}}.$$

Since $|cz + d|^{-2} |dz| = |d(\gamma z)|$, the first of these two integrals is the length of the segment $[R_{x,N}, S_{x,N}]$ and the second the length of the arc of circle $\gamma^{-1}([R_{x,N}, S_{x,N}])$. The first one is bounded by $\frac{2\sqrt{2}}{cN}$ and the second one by $2\pi \frac{N^2}{c^2}$ by corollary to proposition 5. Hence we have

$$\int_{\gamma^{-1}([R_{x,N}, S_{x,N}])} |cz + d|^{k-2} |dz| \leq \left(\frac{2\sqrt{2}}{cN} \right)^{1-\frac{k}{2}} \left(2\pi \frac{N^2}{c^2} \right)^{\frac{k}{2}} \leq 2\pi N^{\frac{3k}{2}-1} c^{-\frac{k}{2}-1}.$$

This completes the proof.

3.4. Second error term: comparing a segment to a Hankel contour

We keep the notations of subsection 3.2. Moreover, we write $j(\gamma, z) = \varepsilon(\gamma) \left(\frac{cz+d}{i} \right)^k$. We then have for each $\lambda \in \Lambda$,

$$\begin{aligned} & \int_{n+i}^{-m+i} j(\gamma, z) P(z) e^{-2\pi i \lambda \gamma z} d(\gamma z) \\ (32) \quad &= \sum_{\mu \in \Lambda, \mu < 0} a(\mu) \varepsilon(\gamma) \int_{-m+i}^{n+i} \left(\frac{cz+d}{i} \right)^{k-2} e^{2\pi i (\mu z - \lambda \frac{az+b}{cz+d})} dz \\ &= \sum_{\mu \in \Lambda, \mu < 0} a(\mu) \varepsilon(\gamma) e^{-2\pi i \frac{\lambda a + \mu d}{c}} \int_{c-ic'}^{c+ic''} v^{k-2} e^{\frac{2\pi}{c} (\mu |v| + \frac{\lambda}{v})} \frac{dv}{ic}, \end{aligned}$$

the last equality being obtained by the change of variable $v = \frac{cz+d}{i}$. In the next proposition L_{1-k} denotes the function defined in subsection 1.2.

Proposition 7.— Assume $\lambda \geq 0$. The integral $\int_{c-ic'}^{c+ic''} v^{k-2} e^{\frac{2\pi}{c}(\mu|v+\frac{\lambda}{v})} \frac{dv}{ic}$ differs from

$$(2\pi)^{2-k} c^{k-2} |\mu|^{1-k} L_{1-k} (4\pi^2 \lambda |\mu| c^{-2})$$

by at most $e^{4\pi\lambda/(N+1)^2} N^{k-2} g_k(\mu)$, where

$$(33) \quad g_k(\mu) = \begin{cases} 2^{k/2} \left(e^{2\pi|\mu|} \left(1 + \frac{1}{2\pi|\mu|} \right) + \frac{\Gamma(k-1)}{(2\pi|\mu|)^{k-1}} \right) & \text{if } k \geq 2, \\ e^{2\pi|\mu|} \left(4 + \frac{1}{\pi|\mu|} \right) & \text{if } 0 \leq k \leq 2. \end{cases}$$

Let H denote the Hankel contour $H_{c,N}$ defined in lemma 1, b). By the change of variable $w = \frac{2\pi|\mu|v}{c}$ and the remark of subsection 1.2, we get

$$\begin{aligned} \int_H v^{k-2} e^{\frac{2\pi}{c}(\mu|v+\frac{\lambda}{v})} \frac{dv}{ic} &= (2\pi)^{2-k} c^{k-2} |\mu|^{1-k} \int_{\frac{2\pi|\mu|}{c}H} w^{k-2} e^{\left(w + \frac{4\pi^2\lambda|\mu|}{c^2w}\right)} \frac{dw}{2\pi i} \\ &= (2\pi)^{2-k} c^{k-2} |\mu|^{1-k} L_{1-k} (4\pi^2 \lambda |\mu| c^{-2}). \end{aligned}$$

In order to prove the proposition, it therefore suffices to prove that the integral of $|v^{k-2} e^{\frac{2\pi}{c}(\mu|v+\frac{\lambda}{v})} \frac{dv}{c}|$ on the complement H' of $[c-ic', c+ic'']$ in H is bounded by $e^{4\pi\lambda/(N+1)^2} N^{k-2} g_k(\mu)$.

We shall first prove that we have $\operatorname{Re}(\frac{1}{v}) = \frac{s}{s^2+t^2} \leq \frac{2c}{(N+1)^2}$ for $v = s + it$ on H' . When $s = c$ and $t \in [c'', N]$, our assertion follows from the inequality $\frac{c}{c^2+t^2} \leq \frac{c}{c^2+c''^2} \leq \frac{2c}{(N+1)^2}$. When $s = c$ and $t \in [-N, -c']$, we argue similarly. When $s \in [0, c]$ and $t = \pm N$, our assertion follows from the inequality $\frac{s}{s^2+N^2} \leq \frac{c}{c^2+N^2} \leq \frac{c}{c^2+c'^2} \leq \frac{2c}{(N+1)^2}$. Finally our assertion is obvious when $s \leq 0$. Hence we have $|e^{2\pi\frac{\lambda}{cv}}| = e^{2\pi\frac{\lambda}{c}\operatorname{Re}(\frac{1}{v})} \leq e^{4\pi\lambda/(N+1)^2}$ for any v on H' . Therefore $\int_{H'} |v^{k-2} e^{\frac{2\pi}{c}(\mu|v+\frac{\lambda}{v})} \frac{dv}{c}| \leq e^{4\pi\lambda/(N+1)^2} J$, where $J = \int_{H'} |v|^{k-2} e^{\frac{2\pi}{c}|\mu|\operatorname{Re}(v)} \frac{|dv|}{c}$.

We shall now bound the integral J . The segment $[c+ic'', c+iN]$ has length at most c since we have $c+c'' > N$. On this segment, we have $|v| \leq N\sqrt{2}$ and $|v| \geq \sqrt{c^2+c''^2} \geq \frac{N}{\sqrt{2}}$. Hence the contribution of this segment to the integral J is at most $e^{2\pi|\mu|} (N\sqrt{2})^{k-2}$ when $k \geq 2$ and $e^{2\pi|\mu|} (\frac{N}{\sqrt{2}})^{k-2}$ when $0 \leq k \leq 2$. The same holds for the segment $[c-iN, c-ic']$.

For $v = s + iN$ with s in $]-\infty, c]$, we have $|v| \geq N$, $|v| \leq N\sqrt{2}$ if $s \in [0, c]$ and $|v| \leq \sup(N\sqrt{2}, |s|\sqrt{2})$ if $s \leq 0$. Hence the contribution of the half line $]-\infty + iN, c + iN]$ to the integral J is bounded by

$$\begin{aligned} &\int_{-\infty}^c (N\sqrt{2})^{k-2} e^{\frac{2\pi}{c}|\mu|s} \frac{ds}{c} + \int_{-\infty}^0 (|s|\sqrt{2})^{k-2} e^{\frac{2\pi}{c}|\mu|s} \frac{ds}{c} \\ &= (N\sqrt{2})^{k-2} \frac{e^{2\pi|\mu|}}{2\pi|\mu|} + \frac{\Gamma(k-1)}{(2\pi|\mu|)^{k-1}} (c\sqrt{2})^{k-2} \end{aligned}$$

when $k \geq 2$ and by

$$\int_{-\infty}^c N^{k-2} e^{\frac{2\pi}{c}|\mu|s} \frac{ds}{c} = N^{k-2} \frac{e^{2\pi|\mu|}}{2\pi|\mu|}$$

when $0 \leq k \leq 2$. The same holds for the other half line $] -\infty - iN, c - iN]$. This completes the proof.

3.5. An approximate value of the coefficients $a(\lambda)$ for $\lambda \geq 0$ in Λ

Assumptions and notations are those of subsection 3.0. In order to state the main result of this subsection, it will be convenient to define two constants which depend only on the modular form f and neither on λ nor on N . The first one is $M_1(f) = \|f - P\|_{\overline{S}_1}$, where P is the polar part of f (see subsection 3.3). The second one is $M_2(f) = \sum_{\mu \in \Lambda, \mu < 0} |a(\mu)| g_k(\mu)$, where the real numbers $g_k(\mu)$ are defined in proposition 7.

Proposition 8.— *For each $\lambda \geq 0$ in Λ and each integer $N \geq 1$, we have*

$$\begin{aligned} & \left| a(\lambda) - (2\pi)^{2-k} \sum_{c=1}^N c^{k-2} \sum_{\mu \in \Lambda, \mu < 0} a(\mu) |\mu|^{1-k} A_c(\lambda, \mu) L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}) \right| \\ & \leq e^{4\pi\lambda/(N+1)^2} \left(\pi M_1(f) \times \begin{cases} (2N^2)^{k-1} \zeta(k-1) & \text{if } k > 2 \\ 2N^2(1 + \log N) & \text{if } k = 2 \\ \frac{4}{2-k} N^k & \text{if } 0 \leq k < 2 \end{cases} + M_2(f) N^k \right), \end{aligned}$$

where the numbers $A_c(\lambda, \mu)$ are defined in subsection 1.4.

In the expression (29) of $a(\lambda)$, we replace the integrals by those given in the left hand side of (30). We then replace f by its polar part and get an error term (31) that we have to sum up when x runs over a system of representatives of \mathbf{Q}_N modulo \mathbf{Z} , for example the rational numbers $\frac{a}{c}$ where $1 \leq c \leq N$, $0 \leq a < c$ and $\gcd(a, c) = 1$. We note that to each $c \geq 1$ correspond at most c values of a . Each integral is then evaluated using (32) and proposition 7. We thus get its approximate value with an error term that we have to sum up over the same system of representatives of \mathbf{Q}_N modulo \mathbf{Z} . By following this procedure we get the following bound for the left hand side in proposition 8:

$$\begin{aligned} & \pi e^{4\pi\lambda/(N+1)^2} \|f - P\|_{\overline{S}_1} \times \begin{cases} (2N^2)^{k-1} \sum_{c=1}^N c^{1-k} & \text{if } k \geq 2 \\ 2N^{\frac{3k}{2}-1} \sum_{c=1}^N c^{-\frac{k}{2}} & \text{if } 0 \leq k \leq 2 \end{cases} \\ & + e^{4\pi\lambda/(N+1)^2} N^{k-2} \sum_{\mu \in \Lambda, \mu < 0} \sum_{c=1}^N c |a(\mu)| g_k(\mu). \end{aligned}$$

We can slightly simplify this expression by using the following inequalities

$$\sum_{c=1}^N c^{1-k} \leq \begin{cases} \zeta(k-1) & \text{if } k > 2 \\ 1 + \log N & \text{if } k = 2, \end{cases}$$

$$\sum_{c=1}^N c^{-\frac{k}{2}} \leq \int_0^N t^{-\frac{k}{2}} dt = \frac{2}{2-k} N^{1-\frac{k}{2}} \quad \text{if } 0 \leq k < 2,$$

$$\sum_{c=1}^N c \leq N^2.$$

Proposition 8 follows.

Corollary 1.— *For each $\lambda \geq 1$ in Λ , we have*

$$\left| a(\lambda) - (2\pi)^{2-k} \sum_{c=1}^{[\sqrt{\lambda}]} c^{k-2} \sum_{\mu \in \Lambda, \mu < 0} a(\mu) |\mu|^{1-k} A_c(\lambda, \mu) L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}) \right|$$

$$\leq e^{4\pi} \left(\pi M_1(f) \times \begin{cases} (2\lambda)^{k-1} \zeta(k-1) & \text{if } k > 2 \\ \lambda(2 + \log \lambda) & \text{if } k = 2 \\ \frac{4}{2-k} \lambda^{k/2} & \text{if } 0 \leq k < 2 \end{cases} + M_2(f) \lambda^{k/2} \right).$$

We apply proposition 8 with $N = [\sqrt{\lambda}]$. We note that in this case, we have $N + 1 \geq \sqrt{\lambda}$, hence $e^{4\pi \lambda / (N+1)^2} \leq e^{4\pi}$.

Remark.— The exponent $k-1$ of λ in the error term of corollary 1 for $k > 2$ is sharp. Indeed, there exist Eisenstein series of weight k with no polar part for which $|a(\lambda)|$ grows as fast as λ^{k-1} when $\lambda \rightarrow +\infty$.

3.6. Proof of theorem 3 when $\Gamma = \text{SL}_2(\mathbf{Z})$

Assumptions and notations are as in subsection 3.0. We recall that the weight k of f is assumed to be ≥ 0 . In this section we furthermore assume that f is not finite at ∞ . This means that there exists at least one $\mu < 0$ in Λ such that $a(\mu) \neq 0$. Let μ_0 be the one with largest absolute value. Let $\delta = |\mu_0|$ and δ' denote the second largest of the numbers $|\mu| c^{-2}$, where μ is a negative element in Λ such that $a(\mu) \neq 0$ and c is a positive integer.

As we noticed in remark 2 of subsection 1.4, $j\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z\right)$ can be written as $\varepsilon \left(\frac{z}{i}\right)^k$ for some ε equal to 1 or -1 and we have $A_1(\lambda, \mu) = \varepsilon$ for any pair $(\lambda, \mu) \in \Lambda^2$.

Proposition 9.— *The coefficient $a(\lambda)$ is equivalent to*

$$(2\pi)^{2-k} a(\mu_0) \delta^{1-k} \varepsilon L_{1-k}(4\pi^2 \lambda \delta)$$

when $\lambda \in \Lambda$ goes to $+\infty$.

The previous expression is nothing but the term corresponding to $(\mu, c) = (\mu_0, 1)$ in the double sum appearing in the left hand side of corollary 1 to proposition 8. The error term on the right hand side is bounded by a power of λ when λ goes to ∞ , hence is negligible with respect to $L_{1-k}(4\pi^2\lambda\delta)$. It therefore suffices to show that the double sum

$$(34) \quad \sum_{c, \mu} a(\mu) c^{k-2} |\mu|^{1-k} A_c(\lambda, \mu) L_{1-k}(4\pi^2\lambda|\mu|c^{-2}),$$

where c runs from 1 to $[\sqrt{\lambda}]$ and μ over the negative elements of Λ such that $a(\mu) \neq 0$, with $(\mu, c) \neq (\mu_0, 1)$, is also negligible with respect to $L_{1-k}(4\pi^2\lambda\delta)$ when λ goes to ∞ .

The function L_{1-k} defined in subsection 1.2 by $L_{1-k}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! \Gamma(n+2-k)}$ may have some negative coefficients, but its coefficients for $n \geq k-2$ are ≥ 0 . We shall therefore introduce another function \tilde{L}_{1-k} defined by

$$\tilde{L}_{1-k}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! |\Gamma(n+2-k)|}.$$

For $t \in \mathbf{R}_+$, we have $|L_{1-k}(t)| \leq \tilde{L}_{1-k}(t)$. Moreover, \tilde{L}_{1-k} is an increasing function on \mathbf{R}_+ and $L_{1-k}(t)$ is equivalent to $\tilde{L}_{1-k}(t)$ when t goes to ∞ .

These properties and the inequalities $|A_c(\lambda, \mu)| \leq c$ and $\sum_{c=1}^{[\sqrt{\lambda}]} c^{k-1} \leq \lambda^{k/2}$ allow us to bound the absolute value of (34) by

$$\lambda^{k/2} \sum_{\mu \in \Lambda, \mu < 0} a(\mu) |\mu|^{1-k} \tilde{L}_{1-k}(4\pi^2\lambda\delta').$$

To complete the proof, we need only to show that $\lambda^{k/2} \tilde{L}_{1-k}(4\pi^2\lambda\delta')$, or equivalently $\lambda^{k/2} L_{1-k}(4\pi^2\lambda\delta')$, is negligible with respect to $L_{1-k}(4\pi^2\lambda\delta)$. But this follows from the fact that $L_{1-k}(t)$ is equivalent to $\frac{1}{\sqrt{4\pi}} t^{\frac{2k-3}{4}} e^{2\sqrt{t}}$ when $t \rightarrow +\infty$.

Corollary.— *There exists $\lambda_0 \in \mathbf{R}_+$ such that $a(\lambda) \neq 0$ for all $\lambda \geq \lambda_0$ in Λ .*

If needed, such a λ_0 could be determined effectively by using the explicit bounds from corollary 1 to proposition 8.

§4. THE CASE OF AN ARBITRARY FINITE INDEX SUBGROUP OF $\mathbf{SL}_2(\mathbf{Z})$

4.0. Assumptions and notations

In this section, we shall indicate how to extend the results of the previous sections when in place of $\mathbf{SL}_2(\mathbf{Z})$ one considers one of its finite index subgroups Γ . So let f be a non-zero modular function for Γ of some real weight k ,

holomorphic on \mathfrak{H} , and let $j(\gamma, z)$ for γ in Γ denote its automorphy factors. We have

$$f(\gamma z) = j(\gamma, z)f(z) \quad \text{and} \quad |j(\gamma, z)| = |cz + d|^k$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ and z in \mathfrak{H} . The functions $j(\gamma, z)$ satisfy the cocycle relation $j(\gamma\gamma', z) = j(\gamma, \gamma'z)j(\gamma', z)$ for γ, γ' in Γ and z in \mathfrak{H} .

We denote by h the smallest positive integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$. The function $j(\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, z)$ takes a constant value u . We have $|u| = 1$ and $f(z + h) = uf(z)$ for $z \in \mathfrak{H}$. We write

$$f(z) = \sum_{\lambda \in \Lambda} a(\lambda) e^{2\pi i \lambda z}$$

the expansion of f at ∞ , where Λ is the set of $\lambda \in \mathbf{R}$ such that $e^{2\pi i \lambda h} = u$. For each λ in Λ , we have

$$(35) \quad a(\lambda) = h^{-1} \int_{[i, i+h]} f(z) e^{-2\pi i \lambda z} dz.$$

4.1. Further notations

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\mathbf{SL}_2(\mathbf{Z})$ with $c > 0$. Define f_γ to be the function $z \mapsto (\frac{cz+d}{i})^{-k} f(\gamma z)$. It is a modular function of weight k for $\gamma^{-1}\Gamma\gamma$, with respect to some automorphy factors. We denote by h_γ the smallest positive integer such that $f_\gamma(z + h_\gamma) = u_\gamma f_\gamma(z)$ for some u_γ in \mathbf{C} . We have $|u_\gamma| = 1$. We denote by $\sum_{\lambda \in \Lambda_\gamma} a_\gamma(\lambda) e^{2\pi i \lambda z}$ the expansion of f_γ at ∞ , where Λ_γ is the set of λ in \mathbf{R} such that $e^{2\pi i \lambda h_\gamma} = u_\gamma$, and by P_γ its polar part $\sum_{\lambda \in \Lambda_\gamma, \lambda < 0} a_\gamma(\lambda) e^{2\pi i \lambda z}$.

We make some elementary remarks on the dependence of all these data on γ :

a) Let $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be another matrix in $\mathbf{SL}_2(\mathbf{Z})$ with $c' > 0$, such that $\Gamma\gamma = \Gamma\gamma'$. Since f is a modular function of weight k for Γ , $f_{\gamma'}$ differs from f_γ by a multiplicative scalar of absolute value 1. Therefore $|f_\gamma|$, $|f_\gamma - P_\gamma|$, h_γ , u_γ , Λ_γ and the numbers $|a_\gamma(\lambda)|$ for $\lambda \in \Lambda_\gamma$ depend only on the coset $\Gamma\gamma$.

b) In particular, let $\gamma' = \begin{pmatrix} 1 & mh \\ 0 & 1 \end{pmatrix} \gamma$ for some $m \in \mathbf{Z}$. We have $f_{\gamma'} = u^m f_\gamma$ and therefore $a_{\gamma'}(\lambda) = u^m a_\gamma(\lambda)$ for all $\lambda \in \Lambda_\gamma$.

c) Let $n \in \mathbf{Z}$ and $\gamma' = \gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. We have $f_{\gamma'}(z) = f_\gamma(z + n)$, therefore we have $h_{\gamma'} = h_\gamma$, $u_{\gamma'} = u_\gamma$, $\Lambda_{\gamma'} = \Lambda_\gamma$ and $a_{\gamma'}(\lambda) = e^{2\pi i \lambda n} a_\gamma(\lambda)$ for $\lambda \in \Lambda_\gamma$.

4.2. Extension of the circle method to Γ

The following proposition generalizes proposition 1 of subsection 1.1.

Proposition 10.— *Assumptions and notations are those of subsection 4.0. We furthermore assume the weight k of f to be < 0 . For each $\lambda \in \Lambda$, we have*

$$a(\lambda) = h^{-1} \sum_x \int_{C_x} f(z) e^{-2\pi i \lambda z} dz,$$

where x runs over a system of representatives of \mathbf{Q} modulo $h\mathbf{Z}$ and C_x is the Ford circle associated to x . Moreover, the integrals and the series in this expression converge absolutely.

The proof is similar to the proof of proposition 1 with the two following minor changes:

- a) One starts with formula (35) in place of formula (5).
- b) For each $x \in \mathbf{Q}$, one chooses a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ such that $c > 0$ and $x = \frac{a}{c}$. Formula (6) gets then replaced by

$$\int_{C_x} |f(z)e^{-2\pi i \lambda z}| |dz| = \int_{i+\mathbf{R}} |cz + d|^{k-2} |f_\gamma(z)e^{-2\pi i \lambda \gamma z}| |dz|.$$

and one notes that the function f_γ is bounded on $i + \mathbf{R}$ and depends only on the coset $\Gamma\gamma$ by subsection 4.1, assertion a).

Let $\lambda \in \Lambda$ and $x \in \mathbf{Q}$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\mathbf{SL}_2(\mathbf{Z})$ such that $x = \frac{a}{c}$ and $c > 0$. As in subsection 1.3, one proves successively the following equalities

$$\begin{aligned} \int_{C_x} f(z)e^{-2\pi i \lambda z} dz &= \int_{i+\infty}^{i-\infty} f(\gamma z)e^{-2\pi i \lambda \gamma z} d(\gamma z) \\ &= \int_{i-\infty}^{i+\infty} \left(\frac{cz+d}{i}\right)^{k-2} f_\gamma(z)e^{-2\pi i \lambda \gamma z} dz = \int_{i-\infty}^{i+\infty} \left(\frac{cz+d}{i}\right)^{k-2} P_\gamma(z)e^{-2\pi i \lambda \gamma z} dz \\ &= \sum_{\mu \in \Lambda_\gamma, \mu < 0} a_\gamma(\mu) \int_{i-\infty}^{i+\infty} \left(\frac{cz+d}{i}\right)^{k-2} e^{2\pi i(\mu z - \lambda \frac{az+b}{cz+d})} dz \\ &= (2\pi)^{2-k} c^{k-2} \sum_{\mu \in \Lambda_\gamma, \mu < 0} a_\gamma(\mu) e^{-2\pi i \frac{\lambda a + \mu d}{c}} |\mu|^{1-k} L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}). \end{aligned}$$

It follows from assertion c) in subsection 4.1 that the number $a_\gamma(\mu)e^{-2\pi i \frac{\lambda a + \mu d}{c}}$ depends only on x and not on the choice of γ . We shall therefore denote it by $\beta_x(\lambda, \mu)$. Since Λ_γ too depends only on x , it is appropriate to denote it by Λ_x . It follows from assertion b) in subsection 4.1 that $\beta_x(\lambda, \mu)$ and Λ_x depend only on x modulo $h\mathbf{Z}$.

Replacing the integrals in proposition 10 by the previous expressions, we get the following generalization of theorem 4:

Theorem 5.— *Let $\lambda \in \Lambda$. With the previous notations, we have*

$$a(\lambda) = (2\pi)^{2-k} h^{-1} \sum_x c^{k-2} \sum_{\mu \in \Lambda_x, \mu < 0} \beta_x(\lambda, \mu) |\mu|^{1-k} L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}),$$

where x runs over the rational numbers in $[0, h[$ and c denotes the denominator of x .

4.3. Proof of theorem 2

For each $\alpha > 0$, let Φ_α denote the set of pairs (x, μ) with the following properties:

a) x is a rational number in $[0, h[$ and μ is a negative element in Λ_x such that $|\mu|c^{-2} = \alpha$, where $c \geq 1$ denotes the denominator of x ;

b) if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathbf{SL}_2(\mathbf{Z})$ such that $x = \frac{a}{c}$, then $a_\gamma(\mu) \neq 0$.

We note that condition b) does not depend on the choice of the matrix γ : this follows from assertion c) in subsection 4.1.

Each set Λ_x has only finitely many negative elements and Λ_x depends only on the orbit Γx of x under Γ , by assertion a) of subsection 4.1. Therefore each set Φ_α is finite and $\{\alpha \in \mathbf{R}_+^* | \Phi_\alpha \neq \emptyset\}$ is a closed discrete bounded subset of \mathbf{R}_+^* . It is non-empty, otherwise f would be equal to 0 by theorem 5; it is in fact infinite. Let δ denote its largest and δ' its second largest element.

Lemma 5.— *There exists λ in Λ such that the complex number*

$$B(\lambda) = \sum_{(x, \mu) \in \Phi_\delta} c^{k-2} \beta_x(\lambda, \mu) |\mu|^{1-k},$$

where c denotes the denominator of x , is different from 0.

Let $\lambda_1 \in \Lambda$. Then Λ is equal to $\lambda_1 + \mathbf{Z}h^{-1}$. For $\lambda = \lambda_1 + nh^{-1}$ in Λ , we have

$$(36) \quad B(\lambda) = \sum_{(x, \mu) \in \Phi_\delta} c^{k-2} \beta_x(\lambda_1, \mu) |\mu|^{1-k} e^{-2\pi i n x / h}.$$

At least one of the coefficients $\beta_x(\lambda_1, \mu)$ appearing in this expression is different from 0, by definition of δ . Furthermore the characters $n \mapsto e^{-2\pi i n x / h}$ of \mathbf{Z} , for $x \in [0, h[\cap \mathbf{Q}$, are distinct, hence linearly independent over \mathbf{C} . Therefore the expression (36) cannot vanish for all n in \mathbf{Z} .

Remark.— Let D denote a common denominator of the rational numbers x , for (x, μ) in Φ_δ . It follows from formula (36) that $B(\lambda)$ depends only on λ modulo $D\mathbf{Z}$.

The sum $\sum_{\mu \in \Lambda_\gamma} |a_\gamma(\mu)| |\mu|^{1-k}$, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ with $c > 0$, depends only on the coset $\Gamma\gamma$ by assertion a) of subsection 4.1. Let $M_3(f)$ denote the supremum of these sums. In the same way as we deduced proposition 4 from theorem 4, we deduce the following result from theorem 5:

Proposition 11.— *For each $\lambda \geq 0$ in Λ , we have*

$$\begin{aligned} & \left| a(\lambda) - (2\pi)^{2-k} h^{-1} B(\lambda) L_{1-k}(4\pi^2 \lambda \delta) \right| \\ & \leq (2\pi)^{2-k} \zeta(1-k) M_3(f) L_{1-k}(4\pi^2 \lambda \delta'). \end{aligned}$$

By lemma 5, there exists λ_0 in Λ such that $B(\lambda_0) \neq 0$. By the remark, we have $B(\lambda) = B(\lambda_0)$ for all $\lambda \in \lambda_0 + D\mathbf{Z}$. Since $L_{1-k}(4\pi^2\lambda\delta')/L_{1-k}(4\pi^2\lambda\delta)$ goes to 0 when $\lambda \rightarrow +\infty$, proposition 11 implies that $a(\lambda)$ is equivalent to $(2\pi)^{2-k}h^{-1}B(\lambda)L_{1-k}(4\pi^2\lambda\delta)$ when $\lambda \in \lambda_0 + D\mathbf{Z}$ goes to $+\infty$, hence does not vanish for $\lambda \in \lambda_0 + D\mathbf{Z}$ large enough. This proves theorem 2.

4.4. Proof of theorem 3

Assumptions and notations are those of subsections 4.0 and 4.1. We moreover assume the weight k of f to be ≥ 0 . We fix an integer $N \geq 1$ and denote by \mathbf{Q}_N the set of rational numbers with denominator $\leq N$.

For each $\lambda \in \Lambda$, we have

$$a(\lambda) = h^{-1} \sum_x \int_{C_{x,N}} f(z) e^{-2\pi i \lambda z} dz,$$

where x runs over a system of representatives of \mathbf{Q}_N modulo $h\mathbf{Z}$, and where $C_{x,N}$ denotes the arc of the Ford circle C_x considered in subsection 3.1. The proof of this formula is the same as the proof of (29) except that we start from formula (35) in place of (28).

Let x be an element in \mathbf{Q}_N . Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ such that $x = \frac{a}{c}$ and $c > 0$. As in subsection 3.2, let n and m denote the largest integers in \mathbf{Z} such that $d + nc \leq N$ and $-d + mc \leq N$ respectively. By following the same steps as in subsection 3.3, one successively proves the following results:

(i) We have for each $\lambda \in \Lambda$

$$\begin{aligned} \int_{C_{x,N}} f(z) e^{-2\pi i \lambda z} dz &= \int_{n+i}^{-m+i} f(\gamma z) e^{-2\pi i \lambda \gamma z} d(\gamma z) \\ &= \int_{-m+i}^{n+i} \left(\frac{cz+d}{i} \right)^{k-2} f_\gamma(z) e^{-2\pi i \lambda \gamma z} dz. \end{aligned}$$

(ii) For each $\lambda \in \Lambda$, the integral $\int_{-m+i}^{n+i} \left(\frac{cz+d}{i} \right)^{k-2} f_\gamma(z) e^{-2\pi i \lambda \gamma z} dz$ differs from $\int_{-m+i}^{n+i} \left(\frac{cz+d}{i} \right)^{k-2} P_\gamma(z) e^{-2\pi i \lambda \gamma z} dz$ by at most

$$\pi e^{4\pi\lambda_+/(N+1)^2} \|f_\gamma - P_\gamma\|_{\overline{\mathfrak{S}}_1} \times \begin{cases} (2N^2)^{k-1} c^{-k} & \text{if } k \geq 2, \\ 2N^{\frac{3k}{2}-1} c^{-\frac{k}{2}-1} & \text{if } 0 \leq k \leq 2. \end{cases}$$

(iii) For each $\lambda \in \Lambda$, we have

$$\int_{-m+i}^{n+i} \left(\frac{cz+d}{i} \right)^{k-2} P_\gamma(z) e^{-2\pi i \lambda \gamma z} dz = \sum_{\mu \in \Lambda_\gamma, \mu < 0} \beta_x(\lambda, \mu) \int_{c-ic'}^{c+ic''} v^{k-2} e^{\frac{2\pi}{c}(|\mu|v + \frac{\lambda}{v})} \frac{dv}{ic},$$

where $\beta_x(\lambda, \mu) = a_\gamma(\mu)e^{-2\pi i \frac{\lambda a + \mu d}{c}}$ as in subsection 4.2 and where $c' = d + nc$ and $c'' = -d + mc$ are the denominators of the predecessor and of the successor of x in \mathbf{Q}_N respectively. Moreover proposition 7 yields when $\lambda \geq 0$ the approximate value $(2\pi)^{2-k} c^{k-2} |\mu|^{1-k} L_{1-k}(4\pi^2 \lambda |\mu| c^{-2})$ for the last integral, together with an upper bound for the error term.

The numbers $\|f_\gamma - P_\gamma\|_{\bar{s}_1}$, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z})$ with $c > 0$, depend only on the coset $\Gamma\gamma$ by assertion a) of subsection 4.1. Let $M_4(f)$ denote the supremum of these numbers. We define in a similar way $M_5(f)$ to be the supremum of the numbers $\sum_{\mu \in \Lambda_x, \mu < 0} |a_\gamma(\mu)| g_k(\mu)$, with $g_k(\mu)$ as in proposition 7. Summing up the previous results, we get the following analog of proposition 8, with the same proof.

Proposition 12.— *For each $\lambda \geq 0$ and each integer $N \geq 1$, we have*

$$\begin{aligned} & \left| a(\lambda) - (2\pi)^{2-k} h^{-1} \sum_x c^{k-2} \sum_{\mu \in \Lambda_x, \mu < 0} \beta_x(\lambda, \mu) |\mu|^{1-k} L_{1-k}(4\pi^2 \lambda |\mu| c^{-2}) \right| \\ & \leq e^{4\pi\lambda/(N+1)^2} \left(\pi M_4(f) \times \begin{cases} (2N^2)^{k-1} \zeta(k-1) & \text{if } k > 2 \\ 2N^2(1 + \log N) & \text{if } k = 2 \\ \frac{4}{2-k} N^k & \text{if } 0 \leq k < 2 \end{cases} + M_5(f) N^k \right), \end{aligned}$$

where x runs over a system of representatives of \mathbf{Q}_N modulo $h\mathbf{Z}$ and c denotes the denominator of x .

Define δ , and $B(\lambda)$ for $\lambda \in \Lambda$, as in subsection 4.3. By taking $N = \lfloor \sqrt{\lambda} \rfloor$ in proposition 12 one gets, in exactly the same way as in subsection 3.6, the following analog of proposition 9.

Proposition 13.— *We have*

$$\left| a(\lambda) - (2\pi)^{2-k} h^{-1} B(\lambda) L_{1-k}(4\pi^2 \lambda \delta) \right| = o(L_{1-k}(4\pi^2 \lambda \delta)),$$

when λ goes to $+\infty$.

By lemma 5, there exists λ_0 in Λ such that $B(\lambda_0) \neq 0$ and by the remark of subsection 4.3, there exists an integer $D \geq 1$ such that $B(\lambda) = B(\lambda_0)$ for $\lambda \in \lambda_0 + D\mathbf{Z}$. It then follows from proposition 13 that $a(\lambda) \neq 0$ for $\lambda \in \lambda_0 + D\mathbf{Z}$ large enough. This proves theorem 3.

§5. LACUNARITY IMPLIES HOLOMORPHY ON \mathfrak{H}

Let g be a meromorphic function on the upper-half plane \mathfrak{H} . Suppose that:

- a) g is holomorphic on some half-plane $\mathfrak{H}_r = \{z \in \mathbf{C} | \text{Im}(z) > r\}$;
- b) there exists a real number $h > 0$ and a complex number u such that $|u| = 1$ and $g(z+h) = ug(z)$ for $z \in \mathfrak{H}$;
- c) g has finite order at ∞ .

Proposition 14.— *If the expansion of g at ∞ is lacunary, then g is holomorphic on \mathfrak{H} .*

The expansion of g at ∞ can be written in the form $\sum_{n=0}^{\infty} a(n)e^{2\pi i(\lambda_0+n)z/h}$ for some $\lambda_0 \in \mathbf{R}$. We can replace g by the function $z \mapsto e^{-2\pi i\lambda_0 z}g(hz)$ and we are then reduced to the case where $g(z+1) = g(z)$ for $z \in \mathfrak{H}$ and where the expansion of g at ∞ is $\sum_{n=0}^{\infty} a(n)e^{2\pi inz}$.

In that case, g can be written as $z \mapsto G(e^{2\pi iz})$, where G is a meromorphic function on the open disc $D = \{z \in \mathbf{C} \mid |z| < 1\}$, holomorphic near 0 and with Taylor expansion $\sum_{n=0}^{\infty} a(n)q^n$ at 0.

Suppose that G is not holomorphic on D . Let r denote the smallest absolute value of the poles of G and let $\alpha_1, \dots, \alpha_m$ be the poles of G with absolute value r , repeated with multiplicity. The Taylor expansion of $G \prod_{i=1}^m (q - \alpha_i)$ around 0 is still lacunary. Replacing G by this function, we are reduced to the case where G has only one pole α with absolute value r , of multiplicity 1.

Let β denote the residue of G at this pole. Then $G - \frac{\beta}{q-\alpha}$ has no pole of absolute value r . Therefore its radius of convergence is $> r$. The coefficients of its Taylor series $\sum_{n=0}^{\infty} (a(n) + \frac{\beta}{\alpha^{n+1}}) q^n$ are such that $a(n) + \frac{\beta}{\alpha^{n+1}} = o(r^{-n})$. Hence $a(n)$ is equivalent to $\frac{-\beta}{\alpha^{n+1}}$ when $n \rightarrow +\infty$ and is different from 0 for n large enough. This contradicts the fact that the series $\sum_{n=0}^{\infty} a(n)q^n$ is lacunary.

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(Sanoli Gun) THE INSTITUTE OF MATHEMATICAL SCIENCES, C.I.T CAMPUS, TARAMANI, CHENNAI 600113, INDIA.

(Joseph Oesterlé) INSTITUT DE MATHÉMATIQUES DE JUSSIEU, 175 RUE DU CHEVALERET,
75013-PARIS, FRANCE.

E-mail address, Sanoli Gun: `sanoli@imsc.res.in`

E-mail address, J. Oesterlé: `oesterle@math.jussieu.fr`